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Finite amplitude surface waves in a liquid layer

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An analysis is presented for the interaction of capillary and gravity waves in a liquid layer of finite depth. The method of multiple scales is used to obtain a third-order expansion uniformly valid for all times. Although this expansion is valid for a wide range of wave-numbers, it breaks down at two critical wave-numbers if the liquid depth is larger than $\sqrt{3}/k_c$, $k_c = (\rho g/T)^{\frac{1}{2}}$, where g is the gravitational acceleration, and ρ and T are the liquid density and surface tension, respectively. For a deep liquid, the singularities are at $k_c/\sqrt{2}$ and $k_c/\sqrt{3}$ respectively, as found by Wilton (1915), and Pierson & Fife (1961).

A second-order expansion valid for wave-numbers near the first critical value (corresponding to a wavelength of 2·44 cm in deep water) is obtained. This expansion shows that two different wave profiles could exist at or near the first critical wave-number. One of these profiles is gravity-like while the other is capillary-like.

1. Introduction

The problem under consideration is finite-amplitude waves in a finite-depth liquid adjacent to a gas with negligible density. The liquid is assumed to be inviscid and unlimited in extent. Only periodic travelling waves are considered in the absence of secondary disturbances of any kind.

Finite-amplitude gravity waves were adequately described many years ago by Stokes (1847), Michell (1893), Wilton (1914) and Levi-Civita (1925). Crapper (1957) presented an exact solution to the non-linear equations of motion when surface tension is the only restoring force. He found that the phase speed decreases rather than increases with increasing amplitude as in the case of gravity waves. Moreover, he predicted that capillary waves have profiles that peak or dimple downward, contrary to the case of gravity waves. Schooley (1958) confirmed Crapper's theory by taking high-speed motion pictures of short-fetch, wind-generated, water waves. He obtained pictures of short capillary waves riding just in front of the start of the crests of gravity waves having velocities equal to those of the capillary waves.

Wilton (1915) and Sekerzh-Zenkovich (1956) analyzed the interaction between gravity and capillary waves in a deep liquid using perturbation techniques. Wilton found difficulties in his expansion (higher order terms are unbounded) for the denumerable set of critical wave-numbers, $k_n' = (\rho g/nT)^{\frac{1}{2}}$, where n is an integer greater than 1, g is the gravitational acceleration acting toward the liquid, and ρ and T are the liquid's density and surface tension respectively. He modified his expansion and obtained a bounded solution at the first critical wave-number (k_2' corresponding to a wavelength of 2·44 cm in deep water). Wilton's latter solution shows that two different wave profiles with different phase speeds could exist at the first critical value. At this critical value he predicted conditions of single- and double-dimpled wave profiles. Schooley (1960) observed double-dimpled wave profiles by means of enlarged pictures of short-fetch, wind-generated waves. Moreover, he used pictures to show that triple, quadruple, etc., dimpled wave profiles can also be observed under proper conditions.

Pierson & Fife (1961) obtained a third-order expansion for the interaction of capillary and gravity waves in a deep liquid using the classical perturbation technique formalized by Stoker (1957) for water waves. This expansion is unbounded at the first two critical values. They modified their solution to obtain a first-order expansion (the second-order solution has an undetermined constant) near the first critical value, using the PLK method (Van Dyke 1964). Nayfeh (1969) obtained a second-order expansion at and near the second critical wave-number (corresponding to a wavelength of 2·99 cm in deep water). He predicts that three different wave profiles could exist in this case. One of these waves is gravity-like, having three dimples, while the other two are capillary-like, having five dimples.

The purpose of this paper is to obtain a second-order expansion valid at and near the first critical wave-number, using the method of multiple scales (Nayfeh 1965, 1968). The present analysis extends the results of Wilton (1915) to the case of a finite depth liquid, and for wave-numbers near the first critical wave-number. The present analysis also extends the results of Pierson & Fife (1961) to the case of a finite liquid, and to second order.

2. Mathematical formulation

The liquid is assumed to be inviscid, and to have a finite depth, but to be otherwise unlimited. One face of the liquid is assumed to be adjacent to a solid wall, while the second face is assumed to be adjacent to a gas whose density is negligible compared to that of the liquid. The motion is assumed to be two-dimensional and to start from rest, so that it can be represented by a potential.

Distances and time are made dimensionless using the wave-number k' and the time $(gk')^{-\frac{1}{2}}$, where g is the gravitational acceleration assumed to be acting toward the liquid. A Cartesian co-ordinate system is introduced, such that the x-axis lies in the plane of the undisturbed surface, and the y-axis normal to this surface and directed away from the liquid. In this co-ordinate system, the dimensionless potential function $\phi(x, y, t)$, representing the liquid oscillations, satisfies

$$\nabla^2\phi = 0, \tag{2.1}$$

for $-\infty < x < \infty$ and $-h \leqslant y \leqslant \eta$, where h is the depth of the layer and $\eta(x, t)$ is the elevation of the wave above the undisturbed surface. At the solid interface, the normal velocity vanishes, i.e.

$$\phi_y(x, -h, t) = 0. \tag{2.2}$$

At the liquid-gas interface, the normal component of the liquid velocity must be equal to the normal velocity of the interface itself, i.e.

$$\eta_t - \eta_x \phi_x + \phi_y = 0 \quad \text{at} \quad y = \eta. \tag{2.3}$$

Moreover, the pressure is constant at the interface, and Bernoulli's equation gives

$$\eta - \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) - k^2 \eta_{xx}(1+\eta_x^2)^{-\frac{3}{2}} = 0 \tag{2.4}$$

at $y = \eta$, where

$$k = k'/k_c, \quad k_c = (\rho g/T)^{\frac{1}{2}}. \tag{2.5}$$

Here, ρ and T are the liquid density and surface tension, respectively. The initial conditions are

$$\eta(x,0) = \sum_{n=1}^{\infty} \epsilon^n f_n(x), \quad \eta_t(x,0) = 0. \tag{2.6}$$

The functions $f_n(x)$ will be chosen to yield periodic travelling waves.

An approximate solution to (2.1)–(2.6) is sought for small but finite ϵ, using perturbation techniques. A straightforward perturbation solution would fail for large times due to the appearance of secular terms. To determine a second-order approximate solution valid for large times, the method of multiple scales is employed by introducing the two new variables, $T_1 = \epsilon t$ and $T_2 = \epsilon^2 t$, in addition to the original variable $T_0 = t$. Since ϵ is small, T_1 and T_2 are slow compared to T_0. The functions $\eta(x,t;\epsilon)$ and $\phi(x,y,t;\epsilon)$ are assumed to possess the following uniformly valid expansions for all times:

$$\eta(x,y,t;\epsilon) = \sum_{n=1}^{3} \epsilon^n \eta_n(x, T_0, T_1, T_2) + O(\epsilon^4), \tag{2.7}$$

$$\phi(x,y,t;\epsilon) = \sum_{n=1}^{3} \epsilon^n \phi_n(x, T_0, T_1, T_2) + O(\epsilon^4). \tag{2.8}$$

In order that (2.7) and (2.8) be uniformly valid, η_n/η_0 must be bounded for all T_0, T_1 and T_2. The time derivative is transformed according to

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial}{\partial T_2}. \tag{2.9}$$

Substituting (2.7)–(2.9) into (2.1)–(2.6), expanding, and equating coefficients of equal powers of ϵ to zero, lead to

order ϵ:

$$\nabla^2 \phi_1 = 0, \tag{2.10a}$$

$$\eta_{1,T_0} + \phi_{1,y} = 0 \quad \text{at} \quad y = 0, \tag{2.10b}$$

$$\eta_1 - \eta_{1,T_0} - k^2 \eta_{1,xx} = 0 \quad \text{at} \quad y = 0, \tag{2.10c}$$

$$\phi_{1,y}(x, -h, T_0, T_1, T_2) = 0, \tag{2.10d}$$

$$\eta_1(x,0) = f_1(x), \quad \eta_{1,T_0}(x,0) = 0; \tag{2.10e}$$

order ϵ^2:

$$\nabla^2 \phi_2 = 0, \tag{2.11a}$$

$$\eta_{2,T_0} + \phi_{2,y} = \phi_{1,x}\eta_{1,x} - \phi_{1,yy}\eta_1 - \eta_{1,T_1} \quad \text{at} \quad y = 0, \tag{2.11b}$$

$$\eta_2 - \phi_{2,T_0} - k^2 \eta_{2,xx} = -\tfrac{1}{2}(\phi_{1,x}^2 + \phi_{1,y}^2) + \phi_{1,T_0 y}\eta_1 + \phi_{1,T_1} \quad \text{at} \quad y = 0, \tag{2.11c}$$

$$\phi_{2,y}(x, -h, T_0, T_1, T_2) = 0, \tag{2.11d}$$

$$\eta_2(x,0) = f_2(x), \quad \eta_{2,T_0} = -\eta_{1,T_1}; \tag{2.11e}$$

order ϵ^3:

$$\nabla^2\phi_3 = 0, \tag{2.12a}$$

$$\eta_{3,T_0}+\phi_{3,y} = \phi_{1,x}\eta_{2,x}+(\phi_{1,xy}+\phi_{2,x})\eta_{1,x}-\phi_{2,yy}\eta_1 -\tfrac{1}{2}\phi_{1,yyy}\eta_1^2-\phi_{1,yy}\eta_2-\eta_{1,T_2}-\eta_{2,T_1} \quad \text{at} \quad y=0, \tag{2.12b}$$

$$\eta_3-\phi_{3,T_0}-k^2\eta_{3,xx} = -\phi_{1,x}\phi_{2,x}-\phi_{1,y}\phi_{2,y} -(\phi_{1,xy}\phi_{1,x}+\phi_{1,yy}\phi_{1,y})\eta_1-\tfrac{3}{2}k^2\eta_{1,xx}\eta_{1,x}^2 +\phi_{2,yT_0}\eta_1+\tfrac{1}{2}\phi_{1,yyT_0}\eta_1^2+\phi_{1,yT_0}\eta_2+\phi_{1\,T_2} +\phi_{2,T_1}+\phi_{1,yT_1}\eta_1 \quad \text{at} \quad y=0, \tag{2.12c}$$

$$\phi_{3,y}(x,-h,T_0,T_1,T_2) = 0, \tag{2.12d}$$

$$\eta_3(x,0) = f_3(x), \quad \eta_{3,T_0} = -\eta_{1,T_2}-\eta_{2,T_1}. \tag{2.12e}$$

In the next section, a third-order solution for these equations is given for the special case $f_1(x) = \cos x$. This solution contains two singularities which depend on the depth of the liquid layer. For deep liquid layers, the two singularities occur at $k^2 = \frac{1}{2}$ and $k^2 = \frac{1}{3}$ as found by Wilton (1915), and higher order solutions have singularities at $k^2 = 1/n$ where n is an integer. A third-order expansion valid near the first singularity is obtained in §4.

3. Solution for $f_1(x) = \cos x$

In this case, secular terms do not arise in the second-order solution, and hence the solution can be shown to be independent of T_1. Therefore, the derivatives with respect to T_1 are dropped in this section.

The periodic travelling wave solution of the first-order equations is

$$\eta_1 = \cos\theta, \tag{3.1a}$$

$$\phi_1 = \sigma_0\frac{\cosh(y+h)}{\sinh h}\sin\theta, \tag{3.1b}$$

$$\theta = x+\sigma_0T_0+\beta_2(T_2), \quad \beta_2(0) = 0, \tag{3.1c}$$

$$\sigma_0^2 = (k^2+1)\tanh h. \tag{3.1d}$$

Equations (3.1) determine uniquely the higher-order terms except for the addition of solutions of the corresponding homogeneous equations. A unique solution is determined by requiring the absence of the fundamental ($\cos\theta$) from η_n for all $n \geqslant 2$. Substitution of (3.1) into the right-hand sides of (2.11b) and (2.11c), and solution of the resulting second-order equations give

$$\eta_2 = a_{22}\cos 2\theta, \tag{3.2a}$$

$$\phi_2 = \sigma_0 e_{22}\frac{\cosh 2(y+h)}{\sinh 2h}\sin 2\theta+\frac{\sigma_0^2}{4}(\coth^2 h-1)T_0, \tag{3.2b}$$

$$a_{22} = \frac{\sigma_0^2}{\mu^2-4\sigma_0^2}\left[\tanh 2h-2\coth h-\frac{\tanh 2h}{2\sinh^2 h}\right], \tag{3.2c}$$

$$e_{22} = a_{22}-\tfrac{1}{2}\coth h, \tag{3.2d}$$

$$\mu_2^2 = 2(4k^2+1)\tanh 2h. \tag{3.2e}$$

Equations (3.1) and (3.2) determine the right-hand sides of (2.12*b*) and (2.12*e*), and they become

$$\eta_{3,T_0}+\phi_{3,y} = -(A_{31}\sigma_0-\beta_2')\sin\theta - A_{33}\sigma_0\sin 3\theta, \quad (3.3a)$$

$$\eta_3-\phi_{3,T_0}-k^2\eta_{3,xx} = (E_{31}\sigma_0^2+\sigma_0\beta_2'\coth h)\cos\theta+E_{33}\sigma_0^2\cos 3\theta, \quad (3.3b)$$

$$A_{31} = \tfrac{1}{2}(\coth h+2\coth 2h)\,a_{22}+\tfrac{3}{8}-\tfrac{1}{2}\coth h\coth 2h, \quad (3.3c)$$

$$A_{33} = \tfrac{3}{2}a_{22}\coth h+\tfrac{3}{8}+3e_{22}\coth 2h, \quad (3.3d)$$

$$E_{31} = \frac{3}{8}\frac{k^2}{\sigma_0^2}+\tfrac{1}{2}a_{22}-\tfrac{5}{8}\coth h+(1-\coth h\coth 2h)\,e_{22}, \quad (3.3e)$$

$$E_{33} = -\frac{3}{8}\frac{k^2}{\sigma_0^2}+\tfrac{1}{2}a_{22}+\tfrac{1}{8}\coth h+(3-\coth h\coth 2h)\,e_{22}. \quad (3.3f)$$

The particular solution of (2.12*a*), (2.12*d*), and (3.3) contains secular terms of the form $T_0\sin\theta$ and $T_0\cos\theta$ unless

$$\beta_2' = \frac{\sigma_0}{2}(A_{31}+E_{31}). \quad (3.4)$$

With this condition, the third-order solution becomes

$$\eta_3 = a_{33}\cos 3\theta, \quad (3.5a)$$

$$\phi_3 = (\beta_2'-A_{31}\sigma_0)\frac{\cosh(y+h)}{\sinh h}\sin\theta + \sigma_0(a_{33}-\tfrac{1}{3}A_{33})\frac{\cosh 3(y+h)}{\sinh 3h}\sin 3\theta, \quad (3.5b)$$

$$a_{33} = \frac{3\sigma_0^2(E_{33}\tanh 3h-A_{33})}{\mu_3^2-9\sigma_0^2}, \quad (3.5c)$$

$$\mu_n^2 = n(n^2k^2+1)\tanh nh. \quad (3.5d)$$

The free surface to third order is thus given by (3.6*a*), and the dimensional phase speed to second order is given by (3.6*b*). Also, there would be an appropriate equation for ϕ.

$$\eta = \epsilon\cos\theta+\epsilon^2a_{22}\cos 2\theta+\epsilon^2a_{33}\cos 3\theta, \quad (3.6a)$$

$$c = \left(\frac{Tg}{\rho}\right)^{\frac{1}{4}}\frac{\sigma_0}{\sqrt{k}}\left\{1+\frac{\epsilon^2}{4}\left[(\coth h+4\coth 2h-3\tanh h)\,a_{22} + 3-2\coth h\coth 2h-\frac{3}{4}\frac{k^2}{\sigma_0^2}\tanh h\right]\right\}. \quad (3.6b)$$

For infinite liquid layers (i.e. $h\to\infty$), (3.6) become

$$\eta = \epsilon\cos\theta+\frac{\epsilon^2}{2}\frac{1+k^2}{1-2k^2}\cos 2\theta+\frac{3\epsilon^2}{16}\frac{2k^4+7k^2+2}{(1-2k^2)(1-3k^2)}\cos 3\theta, \quad (3.7a)$$

$$c = \left(\frac{Tg}{\rho}\right)^{\frac{1}{4}}\left(k+\frac{1}{k}\right)^{\frac{1}{2}}\left[1+\frac{\epsilon^2}{16}\frac{8+k^2+2k^4}{(1-2k^2)(1+k^2)}\right]. \quad (3.7b)$$

This solution is in full agreement with those of Wilton (1915), Pierson & Fife (1961) (except for a typographical error in the third-order term), and Nayfeh (1969). As found by Wilton, (3.7*a*) breaks down as $k^2\to\frac{1}{2}$ or $\frac{1}{3}$, and the phase speed breaks down as $k^2 \to \frac{1}{2}$. Higher-order terms break down as $k^2 \to 1/n$, where n is an integer. Equations (3.7) can be shown to reduce to the gravity waves ($k \to 0$) of Stokes to third order (see Lamb 1932; Kinsman 1965), and to the capillary waves ($k \to \infty$) Crapper (1957) to third order.

Although (3.6) are valid for a wide range of wave-numbers, they break down when

$$\mu_n^2 - n^2\sigma_0^2 = 0, \quad n = 2 \text{ and } 3, \tag{3.8a}$$

or

$$n(k^2+1)\tanh k\bar{h} - (n^2k^2+1)\tanh nk\bar{h} = 0, \tag{3.8b}$$

where $\bar{h} = k_c h'$ with h' the dimensional depth of the layer. It can be shown, by carrying out the expansion to higher orders, that the jth term has singularities

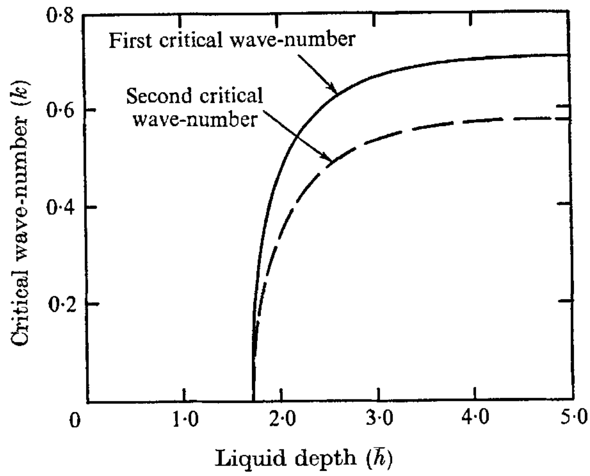

FIGURE 1. Variation of the first two critical wave-numbers with liquid depth.

at the solutions of (3.8b) for $n = 2, 3, \ldots, j$. If $\bar{h} \to \infty$, the solutions of (3.8b) are $k^2 = 1/n$. For small $k\bar{h}$, (3.8b) reduces to

$$k^2 = \frac{15}{(n^2+1)\bar{h}^2}\left(1 - \frac{3}{\bar{h}^2}\right).$$

Thus, the expansion to any order is regular for $\bar{h} < \sqrt{3}$. The solution $k(\bar{h}; n)$ of (3.8b) is shown in figure 1 for $n = 2$ and 3. A second-order expansion valid for wave-numbers near the first singularity (which corresponds to a wavelength of 2·44 cm in deep water) is given in the next section.

4. Solution near first singularity

In order to obtain a valid solution near k_0, where $4\sigma_0^2(k_0) = \mu_2^2(k_0)$, it is assumed that

$$k = k_0 + \epsilon\alpha, \tag{4.1}$$

with $\alpha = O(1)$. Following Wilton (1915), we modify the initial condition

$$f_1(x) = \cos x$$

to contain the first and second harmonics, i.e.

$$f_1(x) = \cos x + b_1 \cos 2x, \tag{4.2}$$

where b_1 is a constant of $O(1)$ which will be determined from the second-order solution. Equation (4.1) leaves (2.10)–(2.12) unchanged, except that (2.11*c*) and (2.12*c*) are modified to include the additive terms, $2k_0\alpha\eta_{1,xx}$ and

$$2k_0\alpha\eta_{2,xx}+\alpha^2\eta_{1,xx},$$

respectively.

The periodic solution to the first-order equations (2.10), subject to the modified initial condition (4.2), is

$$\eta_1 = \cos\theta + b_1\cos 2\theta, \tag{4.3a}$$

$$\phi_1 = \sigma_0\frac{\cosh(y+h)}{\sinh h}\sin\theta+\sigma_0 b_1\frac{\cosh 2(y+h)}{\sinh 2h}\sin 2\theta, \tag{4.3b}$$

$$\theta = x+\sigma_0 T_0+\beta_1(T_1,T_2),\quad \beta_1(0,0)=0. \tag{4.3c}$$

The dependence of β_1 on T_1 as well as the constant b_1 will be determined from the second-order solution.

Substitution for η_1 and ϕ_1 from (4.3) into (2.11*b*) and the modified (2.11*c*) gives

$$\begin{aligned}\eta_{2,T_0}+\phi_{2,y} = &-\left[\frac{b_1}{2}\sigma_0(\coth h+2\coth 2h)-\beta_{1,T_1}\right]\sin\theta\\ &-[\sigma_0\coth h-2b_1\beta_{1,T_1}]\sin 2\theta-\tfrac{3}{2}\sigma_0 b_1(\coth h+2\coth 2h)\\ &\sin 3\theta-4\sigma_0 b_1^2\coth 2h\sin 4\theta,\end{aligned} \tag{4.4a}$$

$$\begin{aligned}\eta_2-\phi_{2,T_0}-k_0^2\eta_{2,xx} = &-q_{20}+[\sigma_0\beta_{1,T_1}\coth h-2k_0\alpha+\tfrac{1}{2}b_1\sigma_0^2(2\coth h\coth 2h-3)]\\ &\times\cos\theta+[2\sigma_0 b_1\beta_{1,T_1}\coth 2h-8k_0\alpha b_1-\tfrac{1}{4}\sigma_0^2(\coth^2 h-3)]\cos 2\theta\\ &-\tfrac{1}{2}\sigma_0^2 b_1(2\coth h\coth 2h-7)\cos 3\theta-b_1^2\sigma_0^2(\coth^2 2h-3)\cos 4\theta,\end{aligned} \tag{4.4b}$$

$$q_{20}=\tfrac{1}{4}\sigma_0^2(\coth 2h-1)+\sigma_0^2 b_1^2(\coth^2 2h-1). \tag{4.4c}$$

The particular solution of (2.11*a*), (2.11*d*), and (4.4) contains secular terms of the form $T_0(\sin\theta,\ \cos\theta,\ \sin 2\theta$, and $\cos 2\theta)$, unless the following conditions are satisfied:

$$\beta_1=\sigma_1 T_1+\beta_2(T_2), \tag{4.5a}$$

$$\sigma_1=\frac{k_0\alpha}{\sigma_0}\tanh h+\tfrac{1}{4}[4\coth 2h+\coth h-3\tanh h]\,\sigma_0 b_1, \tag{4.5b}$$

$$b_1\sigma_1=\frac{2k_0\alpha b_1}{\sigma_0}\tanh 2h+\frac{\sigma_0}{16}[4\coth h+(\coth^2 h-3)\tanh 2h]. \tag{4.5c}$$

With these conditions, the solution of the second-order equations becomes

$$\eta_2=b_2\cos 2\theta+p_{23}\cos 3\theta+p_{24}\cos 4\theta, \tag{4.6a}$$

$$\begin{aligned}\phi_2 = q_{Q0}T_0+\sigma_0 q_{21}&\frac{\cosh(y+h)}{\sinh h}\sin\theta+\sigma_0(q_{22}+b_2)\frac{\cosh 2(y+h)}{\sinh 2h}\sin 2\theta\\ &+\sigma_0 q_{23}\frac{\cosh 3(y+h)}{\sinh 3h}\sin 3\theta+\sigma_0 q_{24}\frac{\cosh 4(y+h)}{\sinh 4h}\sin 4\theta,\end{aligned} \tag{4.6b}$$

where $$p_{23} = -\frac{3\sigma_0^2}{2} b_1 \frac{3(\coth h + 2\coth 2h) + (2\coth h \coth 2h - 7)\tanh 3h}{\mu_3^2 - 9\sigma_0^2}, \tag{4.6c}$$

$$p_{24} = -4\sigma_0^2 b_1^2 \frac{4\coth 2h + (\coth^2 2h - 3)\tanh 4h}{\mu_4^2 - 16\sigma_0^2}, \tag{4.6d}$$

$$q_{21} = -\tfrac{1}{2} b_1 (\coth h + 2\coth 2h) + \sigma_1/\sigma_0, \tag{4.6e}$$

$$q_{22} = -\tfrac{1}{2}\coth h + b_1\sigma_1/\sigma_0, \tag{4.6f}$$

$$q_{23} = [p_{23} - \tfrac{1}{2} b_1 (\coth h + 2\coth 2h)], \tag{4.6g}$$

$$q_{24} = (p_{24} - b_1^2 \coth 2h). \tag{4.6h}$$

The function $\beta_2(T_2)$ and the parameter b_2 will be determined from the third-order equations.

Elimination of σ_1 from (4.5) gives

$$b_1^2 - 2\gamma_1 \frac{k_0\alpha}{\sigma_0^2} b_1 - \gamma_2 = 0, \tag{4.7a}$$

where $$\gamma_1 = \frac{2(2\tanh 2h - \tanh h)}{4\coth 2h + \coth h - 3\tanh h}, \tag{4.7b}$$

$$\gamma_2 = \tfrac{1}{4}\coth h \tanh 2h. \tag{4.7c}$$

The solution of (4.7*a*) is

$$b_1 = \frac{k_0\alpha\gamma_1}{\sigma_0^2} \mp \left[\frac{\alpha^2 k_0^2 \gamma_1^2}{\sigma_0^4} + \gamma_2\right]^{\frac{1}{2}}. \tag{4.8}$$

The above first- and second-order solutions determine the right-hand sides of (2.12*b*) and the modified (2.12*c*). They become

$$\eta_{3,T_0} + \phi_{3,y} = [\beta_2' - (\tfrac{1}{2}\coth h + \coth 2h)\sigma_0 b_2]\sin\theta + 2(b_1\beta_2' + b_2\sigma_1)\sin 2\theta - \sigma_0 \sum_{m=1}^{6} P_{3m}\sin m\theta, \tag{4.9a}$$

$$\eta_3 - \phi_{3,T_0} - k_0^2\eta_{3,xx} = [\beta_2'\sigma_0\coth h + (\tfrac{3}{2} - \coth h\coth 2h)\sigma_0^2 b_2]\cos\theta + 2[\sigma_0 b_1\beta_2'\coth 2h + (\sigma_0\sigma_1\coth 2h - 4k_0\alpha) b_2]\cos 2\theta + \sigma_0^2 \sum_{m=0}^{6} Q_{3m}\cos m\theta, \tag{4.9b}$$

with the P's and Q's given in the appendix. Elimination of the secular terms from the third-order equations necessitates the satisfaction of the following two conditions:

$$2\beta_2' + \gamma_3 b_2 = \lambda_1, \tag{4.10a}$$

$$2\beta_2' + \gamma_4 b_2 = \lambda_2, \tag{4.10b}$$

where
$$\gamma_3 = \sigma_0(3\tanh h - \coth h - 4\coth 2h)/2,$$
$$\gamma_4 = (2\sigma_1\sigma_0 - 4k_0\alpha\tanh 2h)/\sigma_0 b_1,$$
$$\lambda_1 = \sigma_0(P_{31} - Q_{31}\tanh h),$$
$$\lambda_2 = \sigma_0(P_{32} - Q_{32}\tanh 2h)/2b_1.$$

Hence, $$b_2 = (\lambda_1 - \lambda_2)/(\gamma_3 - \gamma_4), \tag{4.11a}$$

and $$\beta_2' = \tfrac{1}{2}(\lambda_1 - \gamma_3 b_2). \tag{4.11b}$$

With these conditions, the solution for η_3 is

$$\eta_3 = b_3 \cos 2\theta + \sum_{m=3}^{6} p_{3m} \cos m\theta, \tag{4.12a}$$

where
$$p_{3m} = \sigma_0^2 m \frac{Q_{3m} \tanh mh - P_{3m}}{\mu_m^2 - m^2 \sigma_0^2}, \tag{4.12b}$$

where b_3 is a constant which can be determined by carrying out the expansion to fourth order. This is not done in this paper, and b_3 remains undetermined.

The free surface to third order is thus given by

$$\eta = \epsilon \cos \theta + (\epsilon b_1 + \epsilon^2 b_2 + \epsilon^3 b_3) \cos 2\theta + \epsilon^2 (p_{23} \cos 3\theta + p_{24} \cos 4\theta) + \epsilon^3 \sum_{m=3}^{6} p_{3m} \cos m\theta, \tag{4.13}$$

and the wave speed is given by

$$c = \left(\frac{Tg}{\rho}\right)^{\frac{1}{4}} \left(k_0 + \frac{1}{k_0}\right)^{\frac{1}{2}} [1 + \epsilon c_1 + \epsilon^2 c_2], \tag{4.14a}$$

where
$$c_1 = \frac{\sigma_1}{\sigma_0} - \frac{\alpha}{2k_0} \tag{4.14b}$$

and
$$c_2 = \frac{\beta_2'}{\sigma_0} + \frac{3}{8}\frac{\alpha^2}{k_0^2}. \tag{4.14c}$$

Limiting cases

For a deep liquid (i.e. $h \to \infty$),

$$\eta = \epsilon \cos \theta + \epsilon (b_1 + \epsilon b_2 + \epsilon^2 b_3) \cos 2\theta - 3\epsilon^2 [b_1 - \epsilon (\tfrac{1}{16} - \tfrac{29}{8} b_1^2 - b_2 + \tfrac{32}{3} k_0 \alpha b_1)] \cos 3\theta - \epsilon^2 [b_1^2 - \epsilon (\tfrac{19}{4} b_1 - 3b_1^3 - 2b_1 b_2 + \tfrac{20}{3} k_0 \alpha b_1^2)] \cos 4\theta + \tfrac{45}{8} \epsilon^3 b_1^2 \cos 5\theta + \tfrac{12}{10} \epsilon^3 b_1^3 \cos 6\theta, \tag{4.15a}$$

where
$$b_1 = \frac{\sqrt{2}}{3} \alpha \mp \left(\frac{2\alpha^2}{9} + \frac{1}{4}\right)^{\frac{1}{2}}, \tag{4.15b}$$

$$b_2 = -\frac{\frac{23}{16} - \frac{7}{4} b_1^2 + \frac{2}{3} k_0 \alpha b_1 - \frac{1}{3}\alpha^2 - (k_0 \alpha / 6b_1)}{1 - (2k_0 \alpha / 3b_1)}. \tag{4.15c}$$

The phase velocity to second order is

$$c = c_0 [1 + \epsilon c_1 + \epsilon^2 c_2], \tag{4.16a}$$

where
$$c_1 = \mp \tfrac{1}{2} (\tfrac{2}{9} \alpha^2 + \tfrac{1}{4})^{\frac{1}{2}}, \tag{4.16b}$$

$$c_2 = \tfrac{1}{2} b_2 + \tfrac{3}{16} - \tfrac{19}{8} b_1^2 + \tfrac{1}{3} k_0 \alpha b_1 + \tfrac{43}{72} \alpha^2. \tag{4.16c}$$

The second-order part of the above solution can be shown to agree with that of Pierson & Fife (1961), except for a typographical error in the second-order, by expressing the latter in dimensionless quantities.

In the case of deep-water waves at a wavelength of 2·44 cm (i.e. $\alpha \equiv 0$), the above solution becomes

$$\eta = \epsilon \cos \theta + \epsilon [\mp \tfrac{1}{2} - \epsilon + \epsilon^2 b_3] \cos 2\theta - \tfrac{3}{2} \epsilon^2 (\mp 1 - \tfrac{5}{16} \epsilon) \cos 3\theta - \epsilon^2 [\tfrac{1}{4} \pm 3\epsilon] \cos 4\theta + \tfrac{45}{32} \epsilon^3 \cos 5\theta \mp \tfrac{3}{20} \epsilon^3 \cos 6\theta. \tag{4.17}$$

The phase velocity becomes

$$c = c_0(1 \mp \tfrac{1}{4}\epsilon - \tfrac{29}{32}\epsilon^2). \tag{4.18}$$

This solution can be shown to agree with the parametric solution of Wilton (1915), after the latter is corrected for typographical errors.

5. Results and conclusions

The results of § 3 show that the solution of the linearized equations is not arbitrary at or near a denumerable set of wave numbers if the liquid depth is larger than $\sqrt{3}/k_c$. Assuming that the first-order solution contains one harmonic only leads to an expansion which is singular at the wave-numbers $k_c/\sqrt{n}$, where n is an integer.

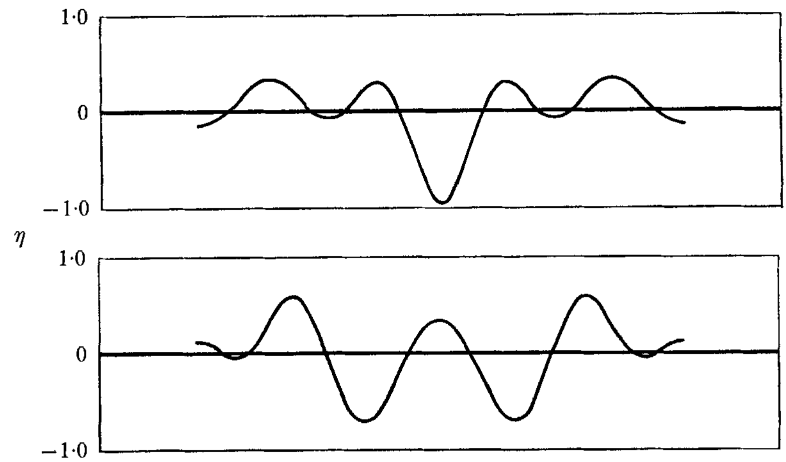

FIGURE 2. The two different wave profiles for a wavelength of 3·73 cm and an amplitude of 0·078 cm in a water layer of depth 0·55 cm. The upper curve is a capillary profile, while the lower curve is a gravity profile. One cycle is shown.

To remove the singularity at the first critical value (corresponding to a wavelength of 2·44 cm in deep water), the first-order solution is assumed to contain the first and second harmonics. The coefficient of the second harmonic is determined from the analysis. The resultant expansion is bounded, and shows that two different wave profiles could exist at or near this critical value, as found by Wilton (1915) for the case of deep water at the first-critical value. One of the profiles is gravity-like having a phase speed that increases while the other is capillary-like having a phase speed that decreases as the amplitude increases.

Figure 2 shows the wave profiles predicted from the present analysis for an amplitude of 0·078 cm at a wavelength of 3·73 cm in a water layer of depth 0·55 cm. The upper curve is capillary-like having three dimples while the lower curve is gravity-like having four dimples.

The upper two curves in figure 3 are the two wave profiles predicted from the present analysis at a wavelength of 2·44 cm in deep water. The upper curve is capillary-like, having one large and three small dimples, while the other is gravity-

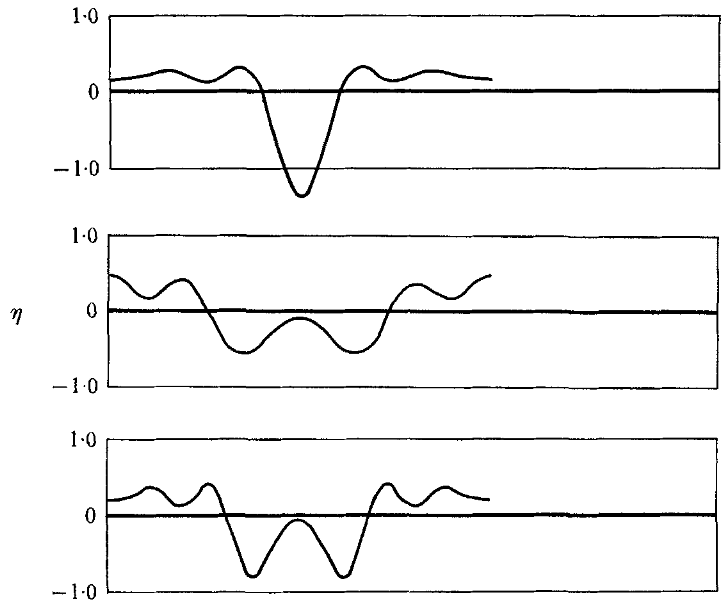

FIGURE 3. Wave profiles in deep water for a wavelength of 2·44 cm and a steepness ratio of 0·4. One cycle is shown. The upper curves are calculated from the present analysis while the lower curve is one of three curves calculated by perturbation from a wavelength of 2·99 cm (Nayfeh 1969).

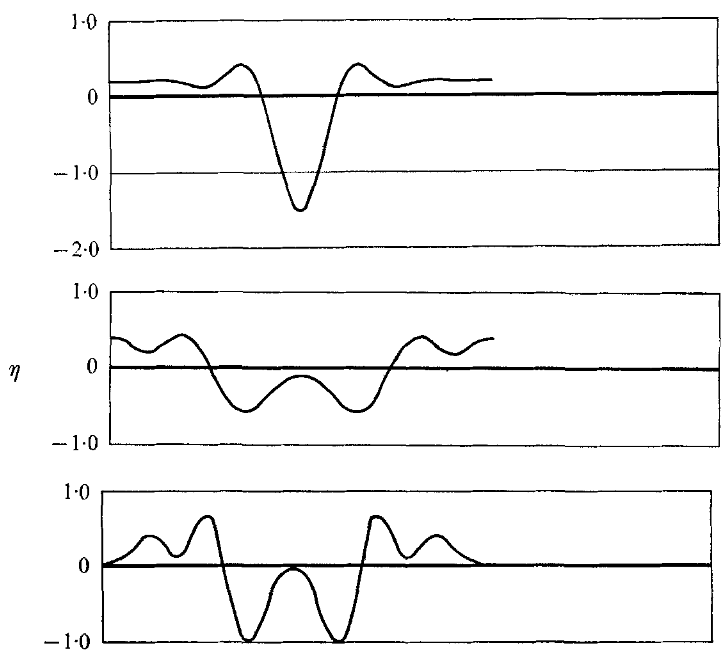

FIGURE 4. Wave profiles in deep water for a wavelength of 2·99 cm and a steepness ratio of 0·4. One cycle is shown. The upper curves are calculated from present analysis by perturbation from a wavelength of 2·44 cm while the lower curve is one of three profiles calculated by Nayfeh (1969).

like, having two large and two small dimples. If one attempts to calculate the wave profiles at a wavelength of 2·44 cm by perturbation from those at a wavelength of 2·99 cm (Nayfeh 1969), he finds that three, rather than two, wave profiles are possible. Of these three profiles, the lower curve in figure 3 is the most qualitatively similar to the profiles predicted from the present analysis. This lower profile is capillary-like, having two large and three small dimples. Figure 3 shows that the analysis of Nayfeh (1969) is not valid at a wavelength of 2·44 cm.

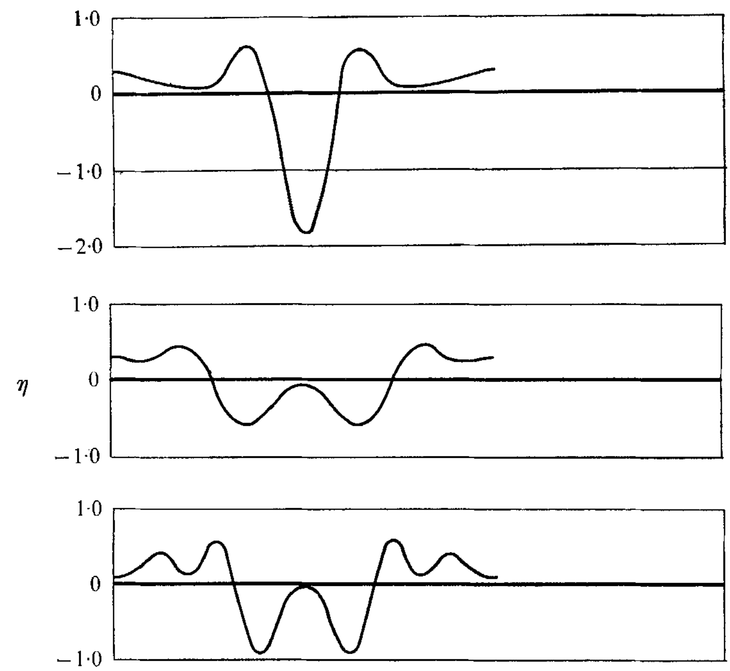

FIGURE 5. Wave profiles in deep water for a wavelength of 2·69 cm. One cycle is shown. The upper curves are calculated by perturbation from a wavelength of 2·44 cm while the lower curve is one of three curves calculated by perturbation from a wavelength of 2·99 cm (Nayfeh 1969).

To determine the range of validity of the present analysis, the two profiles predicted from this analysis at a wavelength of 2·99 cm in deep water are compared in figure 4 with the most qualitatively similar profile of the three profiles predicted by Nayfeh (1969) at a wavelength of 2·99 cm. This comparison shows that the present analysis is not valid at a wavelength of 2·99 cm.

Figure 5 compares three profiles calculated at a wavelength of 2·69 cm in deep water. The upper two profiles are obtained by perturbation from those at a wavelength of 2·44 cm. The third profile in figure 3 is the most qualitatively similar of the three profiles calculated by perturbation from those at a wavelength of 2·99 cm. The difference in these profiles shows that both the present analysis and that of Nayfeh (1969) are valid only in the immediate neighbourhoods of the first and second critical wave-numbers, respectively.

Appendix

If C_m denotes $\coth mh$, then the amplitudes of the forcing functions of (4.4) are

$$
\begin{aligned}
P_{31} &= \tfrac{3}{8} + \tfrac{9}{4}b_1^2 + \tfrac{1}{2}b_1 q_{21} C_1 + (q_{22} + b_1 p_{23}) C_2 + \tfrac{3}{2}b_1 q_{23} C_3, \\
P_{32} &= 3b_1 + 3b_1^3 + (q_{21} + p_{23}) C_1 + 2b_1 p_{24} C_2 + 3q_{23} C_3 + 4b_1 q_{24} C_4, \\
P_{33} &= \tfrac{3}{8} + \tfrac{27}{8}b_1^2 - 3(\sigma_1/\sigma_0) p_{23} + \tfrac{3}{2}(b_1 q_{21} + p_{24} + b_2) C_1 + 3(q_{22} + b_2) C_2 + 6q_{24} C_4, \\
P_{34} &= 3b_1 - 4(\sigma_1/\sigma_0) p_{24} + 2p_{23} C_1 + 4b_1(q_{22} + 2b_2) C_2 + 6q_{23} C_3; \\
P_{35} &= \tfrac{45}{8}b_1^2 + \tfrac{5}{2}p_{24} C_1 + 5b_1 p_{23} C_2 + \tfrac{15}{2}b_1 q_{23} C_3 + 10q_{24} C_4, \\
P_{36} &= 3b_1^3 + 6b_1 p_{24} C_2 + 12b_1 q_{24} C_4, \\
Q_{30} &= 2b_1 b_2 + \tfrac{1}{2}(\sigma_1/\sigma_0)(1 + 4b_1^2) - \tfrac{5}{4}b_1 C_1 - \tfrac{1}{2}b_1 C_2 - \tfrac{1}{2}q_{21} C_1^2 - 2b_1(q_{22} + b_2) C_2^2, \\
Q_{31} &= \tfrac{5}{2}(\sigma_1/\sigma_0) b_1 - (\alpha^2/\sigma_0^2) + 2b_1 p_{23} + q_{22} - \tfrac{1}{2}b_1 q_{21} + \tfrac{3}{2}b_1 q_{23} + 3(b_1^2 + \tfrac{1}{8})(k_0^2/\sigma_0^2) \\
&\quad + \{q_{21}(\sigma_1/\sigma_0) - \tfrac{5}{8} - \tfrac{7}{4}b_1^2\} C_1 - 5b_1^2 C_2 - (b_1 q_{21} + q_{22}) C_1 C_2 - 3b_1 q_{23} C_2 C_3, \\
Q_{32} &= \tfrac{1}{2}(\sigma_1/\sigma_0) - 4(\alpha^2/\sigma_0^2) b_1 + \tfrac{1}{2}p_{23} + 2b_1 p_{24} + 4b_1 q_{24} + 3q_{23} + q_{21} \\
&\quad + 3(2b_1^2 + 1)(k_0^2/\sigma_0^2) b_1 - \tfrac{5}{2}b_1 C_1 + (b_1 - 5b_1^3 + 2(\sigma_1/\sigma_0) q_{22}) C_2 \\
&\quad - \tfrac{1}{2}q_{21} C_1^2 - \tfrac{3}{2}q_{23} C_1 C_3 - 4b_1 q_{24} C_2 C_4, \\
Q_{33} &= \tfrac{5}{2}(\sigma_1/\sigma_0) b_1 - 18(k_0\alpha/\sigma_0^2) p_{23} + \tfrac{1}{2}p_{24} + \tfrac{7}{2}b_2 + 6q_{24} + \tfrac{3}{2}b_1 q_{21} \\
&\quad + 3q_{22} + \tfrac{3}{2}(3b_1^2 - \tfrac{1}{4})(k_0^2/\sigma_0^2) + \tfrac{1}{8}(1 - 11b_1^2) C_1 + \tfrac{1}{2}b_1^2 C_2 \\
&\quad + 3(\sigma_1/\sigma_0) q_{23} C_3 - (b_1 q_{21} + q_{22} + b_2) C_1 C_2 - 2q_{24} C_1 C_4, \\
Q_{34} &= 2(\sigma_1/\sigma_0) b_1^2 - 32(k_0\alpha/\sigma_0^2) p_{24} + 6b_1 b_2 + \tfrac{1}{2}p_{23} + 6q_{23} + 4b_1 q_{22} - 3(k_0^2/\sigma_0^2) b_1 \\
&\quad - \tfrac{1}{4}b_1 C_1 + \tfrac{3}{2}b_1 C_2 - 2b_1(q_{22} + b_2) C_2^2 + 4(\sigma_1/\sigma_0) q_{24} C_4 - \tfrac{3}{2}q_{23} C_1 C_3, \\
Q_{35} &= \tfrac{1}{2}p_{24} + 2b_1 p_{23} + 10q_{24} + \tfrac{15}{2}b_1 q_{23} - \tfrac{15}{2}(k_0^2/\sigma_0^2) b_1^2 - \tfrac{3}{8}b_1^2 C_1 \\
&\quad + \tfrac{5}{2}b_1^2 C_2 - 2q_{24} C_1 C_4 - 3b_1 q_{23} C_2 C_3, \\
Q_{36} &= 2b_1 p_{24} + 12b_1 q_{24} - 6(k_0^2/\sigma_0^2) b_1^3 + C_2 b_1^3 - 4b_1 q_{24} C_2 C_4.
\end{aligned}
$$

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